Optimization Viewpoint of Network Control

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References:

1. Frank Kelly. *Charging and rate control for elastic traffic*
2. Frank Kelly, Aman Maulloo and David Tan. *Rate control for communication networks: shadow prices, proportional fairness and stability.*
3. L. Massoulie and J. Roberts. *Bandwidth sharing: objectives and algorithms*
4. David Lapsley and Steven Low. *An optimization approach to ABR control*

Rate Allocation or Congestion Control

Practitioners

- window-based congestion control: release no more than one extra packets in each round-trip time
- works on short time scale and under transient and random situations
- unsure about achievable rates; no concept of optimal resource allocation

Theoreticians

- deal with optimal rate allocation for some utility functions
- devise algorithms that achieve the allocation
- over-simplified models (so far): assuming traffic is greedy, connections are persistent, and rate is well defined

Problem Formulation

SYSTEM(U,A,C)

$$\begin{aligned} \text{maximize} \quad & \sum_{r \in R} U_r(x_r) \\ \text{subject to} \quad & Ax \leq C \\ \text{over} \quad & x \geq 0 \end{aligned}$$

A: 0-1 matrix. $A_{jr} = 1$ if link j is used by user (route) r.

C: Vector. Link capacities

x: Vector. User's bandwidth

$U_r(x_s)$: Utility of user r as a function of bandwidth. Increasing, strictly concave and continuously differentiable

Kelly's Decomposition

$\mathrm{USER}_r(U_r;\, \lambda_r)$

$$\max_{w_r \geq 0} \quad U_r\left(\frac{w_r}{\lambda_r}\right) - w_r$$

$\mathrm{NETWORK}(A{,}C;\, w)$

$$\max_{x \geq 0} \quad \sum_{r \in R} w_r \log x_r$$

$$\text{subject to} \quad Ax \leq C$$

λ_r**:** Price for user r (per unit bandwidth)

w_r**:** User r's cost or willingness to pay (per unit time)

Note: $x_r = \frac{w_r}{\lambda_r}$

Theorem 1 (Kelly) *There exist vectors* $\lambda = (\lambda_r, r \in R)$, $w = (w_r, r \in R)$ *and* $x = (x_r, r \in R)$ *such that*
(i) w_r *solves* $USER_r(U_r; \lambda_r)$, *for* $r \in R$;
(ii) x *solves NETWORK(A,C; w);*
(iii) $w_r = \lambda_r x_r$ *for* $r \in R$
The vector x *also solves SYSTEM(U,A,C).*

Proof: Write out the Kuhn-Tucker conditions for each of the three optimization problems.

Question: How did Kelly come with the decomposition?

Answer: Not sure. But...

Definition 1 (Proportional Fairness) *A vector of rate $x = (x_r, r \in R)$ is* **proportionally fair** *if it is feasible and if for every other feasible x^*, the aggregate of proportional changes is negative:*

$$\sum_{r \in R} \frac{x_r^* - x_r}{x_r} < 0$$

The solution to NETWORK(A,C; 1) is proportionally fair, because $\nabla f(x)^T(x^* - x) < 0$, for $f(x) = \sum_{r \in R} \log x_r$.

Lapsley and Low Decomposition - Less Magical

$$D(p) := \max_{x \geq 0} L(x,p) = \sum_{r \in R} B_r(\lambda_r) + \sum_l p_l c_l$$

$\mathrm{USER}_r(U_r;\, \lambda_r)$

$$\begin{aligned} B_r(\lambda_r) \quad &:= \quad \max_{x_r \geq 0} U_r(x_r) - \lambda_r x_r \\ &\quad\quad \text{where } \lambda_r = \sum_{l \in r} p_l \end{aligned}$$

NETWORK(A,C; p)

$$\min_{p \geq 0} D(p)$$

Network Algorithm: Gradient Projection

$$p_l(t+1) = (p_l(t) + \gamma(x^l(p(t)) - c_l))^+$$
$$\text{where } \; x^l(p) = \textstyle\sum_{r \ni l} x_r(p)$$

Kelly's Algorithm

$$\frac{d}{dt}x_r(t) = \kappa(w_r - x_r(t)\sum_{l\in r}\mu_l(t)) \qquad \text{for all } r \in R \tag{1}$$

where $\mu_l(t) = \nu_l(\sum_{r\ni l} x_r(t))$.

(1) has a stable equilibrium because

$$V(x) = \sum_{r\in R} w_r \log x_r - \sum_{l\in L}\int_0^{\sum_{r\ni l} x_r} \nu_l(y)dy$$

provides a Lyapunov function.

More generally,

$$V(x) = \sum_{r \in R} U_r(x_r) - \sum_{l \in L} \int_0^{\sum_{r \ni l} x_r} \nu_l(y) dy$$

provides an Lyapunov function for

$$\frac{d}{dt} x_r(t) = \kappa(x_r(t) \dot{U}_r(x_r) - x_r(t) \sum_{l \in r} \mu_l(t)) \qquad \text{for all } r \in R$$

Lyapunov Function

Consider a continuous time system $\dot{x}(t) = f(x(t))$ with an equilibrium point $\bar{x}$.

Definition 2 *A function V is a* **Lyapunov function** *on a region Ω if*
(1) V is continuous and has continuous first partial derivatives;
(2) $V(x)$ has a unique maximum (respectively, minimum) at $\bar{x}$ with respect to all other points in Ω;
(3) The function $\dot{V}(x) = \nabla V(x) f(x)$ satisfies $\dot{V}(x) \geq 0$ (respectively, ≤ 0) for all x in Ω.

Theorem 2 *If there exists a Lyapunov function $V(x)$ in a spherical region $S(\bar{x}, R_o)$ with center $\bar{x}$, then the equilibrium piont $\bar{x}$ is stable.*

Congestion Control

$$dx_r(t) = \kappa(w_r dt - x_r(t) \sum_{l \in r} \varepsilon_l dN_l(\varepsilon_l^{-1} \int_0^t \mu_l(\tau) d\tau)) \qquad (2)$$

where $(N_l(\tau), \tau > 0)$, for $l \in L$ is a collection of independent unit rate Poisson processes.

Modeling:

- Link l generates congestion signals as a time-dependent Poisson process at rate $\varepsilon_l^{-1} \mu_l(t)$;
- a copy of the signal is sent to each user r whose uses link l;
- and user r reduces its rate by an amount $\kappa \varepsilon_l x_r(t)$.

$$((\varepsilon N_l(\tau/\varepsilon)-\tau)\varepsilon^{-1/2},\tau\geq 0)\xrightarrow{d}(B_l(\tau),\tau\geq 0)$$

as $\varepsilon \to 0$, where $(B_l(\tau),\tau \geq 0)$ for $l \in L$ are independent standard Brownian motions. And (2) can be approximated by,

$$dx_r(t)=\kappa(w_r dt - x_r(t)\sum_{l\in r}(\mu_l(t)dt+\varepsilon_l^{1/2}\mu_l(t)^{1/2}dB_l(t))) \qquad (3)$$

Solution to (3) for small ε_l, $l \in L$.

- Ornstein-Uhlenbeck process, centered on the stable point x of the differential equation (1).
- $(x_r(t), r \in R)$ has multivariate normal distribution

Rate Allocation for Window Control Schemes

B_r**:** window size for user (route) r

T_r**:** round-trip time for route r, excluding queueing delay

Theorem 3 *Under fluid model and with* **FIFO** *queueing at each link, for given* B_r *and* T_r, $r \in R$, *the associated rates* x_r, $r \in R$ *are unique solution to the optimization problem*

$$\begin{array}{ll} \max_{x \geq 0} & \sum_{r \in R} B_r \log x_r - x_r T_r \\ \textit{subject to} & Ax \leq C \end{array}$$

Proof: Let $B_{l,r}$ denote route r's backlog at link l. Let $B(l) = \sum_{r \ni l} B_{l,r}$ be total backlog at link l. Use $x_r T_r + \sum_{l \in r} B_{l,r} = B_r$ and $B_{l,r} = x_r \frac{B(l)}{C_l}$, we get $x_r (T_r + \sum_{l \in r} \frac{B(l)}{C_l}) = B_r$, which agrees with the KT conditions.

Maximum Throughput

Theorem 4 *If each link implements per-flow queueing with Longest Queue First policy, for given B_r and T_r, $r \in R$, the associated rates x_r, $r \in R$ are unique solution to the problem*

$$\begin{array}{ll} \max_{x\geq 0} & \displaystyle\sum_{r\in R} B_r x_r - \frac{1}{2}x_r^2 T_r \\ \textit{subject to} & Ax \leq C \end{array}$$

Proof: Let $B(l) = \max_{r\ni l} B_{l,r}$, i.e., the longest queue at link l. The policy is such that if $B_{l,r} < B(l)$, then $x_r = 0$. The Lagrangian is

$$\mathcal{L} = \sum_{r\in R} B_r x_r - \frac{1}{2}x_r^2 T_r + \sum_{l\in L} \mu_l (C_l - \sum_{r\ni l} x_r) + \sum_{r\in R} q_r x_r$$

Kuhn-Tucker condition for the optimization problem gives $x_r T_r = B_r - \sum_{l\in r} \mu_l + q_r$. Identify $\mu_l = B(l)$.

Max-min Fairness

Theorem 5 *If each link implements per-flow fair-queueing policy, for given B_r and T_r, $r \in R$, the associated rates x_r, $r \in R$ are unique and max-min fair with upper bounds B_r/T_r on x_r.*

Proof: $x_r T_r + \sum_{l \in r} B_{l,r} = B_r$ ensures $x_r \leq B_r/T_r$. Also, if $x_r < B_r/T_r$, there exists a link $l \in r$ such that $B_{l,r} > 0$. For this link l, $\sum_{r \in l} x_r = C_l$, since the server is non-idling. Because fair-queueing policy, when $B_{l,r} > 0$, $x_r = \max_{r' \ni l} x_{r'}$. To summarize, for all $r \in R$, $x_r \leq B_r/T_r$, and

$$x_r < B_r/T_r \Rightarrow \text{for some } l \in r, \sum_{r' \ni l} x_{r'} = C_l \text{ and } x_r = \max_{r' \ni l} x_{r'}$$

Minimum Potential Delay

Theorem 6 *If each link implements per-flow queueing with service rate being shared by the queues in proportion to the square roots of the corresponding queue size, then for given B_r and T_r, $r \in R$, the associated rates x_r, $r \in R$ are unique solution to the optimization problem*

$$\begin{array}{ll} \min_{x \geq 0} & \displaystyle\sum_{r \in R} B_r/x_r + T_r \log x_r \\ \textit{subject to} & Ax \leq C \end{array}$$

Proof: Skipped.

Random Search Rate Control Algorithms

- Assume $x_r = \delta \nu_r$, where ν_r is integer-valued, and $0 \le \nu_r \le n_{max}$, δ is fixed.
- ν_r changes in Markovian fashion, jumping from n to $n-1$ with rate d_n and from n to $n+1$ with rate b_n on the condition that this will not lead to capacity being exceeded at some link.

Then, the vector process $(\nu(t), t \ge 0)$ is a reversible Markov process and the stationary distribution is

$$\pi(n_1, n_2, ..., n_K) = \Pi_{r \in R} \frac{b_0 b_1 ... b_{n_r - 1}}{d_1 d_2 ... d_{n_r}} \Pi_{l \in L} \mathbf{1}_{\sum_{r \ni l} \delta n_r \le C_l}$$

Maximum Throughput

Choose $b_n \equiv b$ and $d_n \equiv d$. Then,

$$\pi(n_1, n_2, ..., n_K) = (\frac{b}{d})^{\sum_{r \in R} n_r} \ \Pi_{l \in L} \mathbf{1}_{\sum_{r \ni l} \delta n_r \leq C_l}$$

When b/d becomes large, the stationary distribution concentrates on those rate allocations which maximize the total throughput $\sum_{r \in R} x_r$.

Proportional Fairness

Choose $b_n = (n+1)^\alpha$, $n \geq 0$ and $d_n = (n-1)^\alpha$, for some $\alpha > 0$. Then,

$$\pi(n_1, n_2, ..., n_K) = \exp(\alpha \sum_{r \in R} \log n_r)\ \Pi_{l \in L} \mathbf{1}_{\sum_{r \ni l} \delta n_r \leq C_l}$$

When α becomes increases, the stationary distribution concentrates on those rate allocations which maximize $\sum_{r \in R} \log x_r$.

Max-min Fairness

Choose $b_{n-1}/d_n = \exp A^{M-n}$, where $A > 0$ and $M > \max_{r \in R} n_r$ for any feasible rate allocation $(\delta n_r, r \in R)$. Suppose $(\delta m_r, r \in R)$ is max-min fair and $(\delta n_r, r \in R)$ is not. Then, by definition, there exists a r_o such that $n_{r_o} < m_{r_o}$, and for any other r, either $n_r \le m_r$ or $m_r > n_{r_o}$. Then,

$$\frac{\pi(m_1, m_2, ..., m_K)}{\pi(n_1, n_2, ..., n_K)} \ge \exp(A^{M-m_{r_o}} - K \sum_{i=m_{r_o}+1}^{n_{max}} A^{M-i})$$

When A tends to infinity, this ratio also tends to infinity. Hence, the probability concentrates on the max-min fair rate as $A \to \infty$.

Minimum Potential Delay

Choose

$$\frac{b_{n-1}}{d_n} = \exp(-a(\frac{1}{n} - \frac{1}{n-1})) = \exp \frac{a}{n(n-1)}$$

Then the stationary distribution is proportional to

$$\exp(-a \sum_{r \in R} \frac{1}{n_r}) \ \Pi_{l \in L} \mathbf{1}_{\sum_{r \ni l} \delta n_r \leq C_l}$$

thus concentrates on the feasible allocations which minimize the total potential delay $\sum_{r \in R} 1/x_r$, $a \to \infty$.

Analyzing Deterministic Algorithms

Choose $d_n = n$ and $b_n = n + \alpha/\delta$, where $\alpha > 0$.

- When upward transition is feasible, the drift for x_r is constant and equal to α, producing a linear increase.
- When upward transition is impossible, the drift at point $x = n\delta$ is $-x$, producing an exponential decay.
- In the limit $\delta \to 0$, the rates evolve continuously in deterministic fashion according to additive increase/multiplicative decrease mechanism.

Consider two feasible rate vectors $x = (x_r) = (\delta n_r)$ and $y = (y_r) = (\delta m_r)$. Then

$$\frac{\pi(x)}{\pi(y)} = \exp \sum_{r \in R} \Big(\sum_{j=x_r/\delta+1}^{(x_r+\alpha)/\delta-1} \log j - \sum_{j=y_r/\delta+1}^{(y_r+\alpha)/\delta-1} \log j \Big)$$

The exponent is recognized as Riemann sums and is equivalent to

$$\frac{1}{\delta}\sum_{r\in R}\left(\int_{x_r}^{x_r+\alpha}\log x dx - \int_{y_r}^{y_r+\alpha}\log x dx\right)$$

In the limit $\delta \to 0$, the distribution π concentrates on the feasible rate which maximize $\sum_{r\in R}\int_{x_r}^{x_r+\alpha}\log x dx$. For small α, $\sum_{r\in R}\int_{x_r}^{x_r+\alpha}\log x dx \approx \alpha\sum_{r\in R}\log x$.

Hence, additive increase/multiplicative decrease mechanism realizes proportional fairness.

Theorem 7 (Masoulie and Roberts) *The deterministic congestion avoidance algorithm with increase and decrease functions f_r and g_r for route $r \in R$ has equilibrium points at those rate allocations at which the function*

$$\sum_{r \in R} \int_0^{x_r} \log \frac{f_r(u) + g_r(u)}{g_r(u)} du$$

is maximal

Application: $g(u) = u$ and $f(u) = u(\exp(\frac{1}{u^2} - 1)$ gives a deterministic algorithm that minimizes total potential delay. For large u, $f(u) \approx 1/u$.

Hence, "logarithmic increase/multiplicative decrease leads to a rate allocation that minimizes total potential delay."